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Symmetries of nonlinear diffusion equations

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Abstract. We give the Lie point symmetry groups of a class of nonlinear diffusion equations.

The Lie point symmetries of the linear diffusion equation

$$\partial u/\partial t = \operatorname{div}(\operatorname{grad} u) = \Delta u = \sum_{i=1}^{n} \partial^2 u/\partial x_i^2 \tag{1}$$

in n space dimensions are well known. We summarise this in the following theorem.

Theorem 1. The linear diffusion equation (1) is invariant under the Lie point symmetry groups which are generated by the following infinitesimal generators:

$$T = \partial/\partial t \qquad X_i = \partial/\partial x_i \qquad U = u\,\partial/\partial u \qquad G_i = t\,\partial/\partial x_i - \tfrac{1}{2}x_i u\,\partial/\partial u$$

$$S = \sum_{i=1}^{n} x_i\,\partial/\partial x_i + 2t\,\partial/\partial t \qquad R_{ij} = x_i\,\partial/\partial x_j - x_j\,\partial/\partial x_i$$

$$C = t\sum_{i=1}^{n} x_i\,\partial/\partial x_i + t^2\,\partial/\partial t - (\tfrac{1}{4}r^2 + \tfrac{1}{2}nt)u\,\partial/\partial u$$

where $i, j = 1, \ldots, n$ and $r^2 = x_1^2 + \ldots + x_n^2$.

In elementary terms, invariance of equation (1) under, for example, G_i is equivalent to the statement that $\mathrm{D}u = 0$ implies

$$\mathrm{D}(t\partial/\partial x_i + \tfrac{1}{2}x_i)u = 0$$

where $\mathrm{D}u = \partial u/\partial t - \Delta u$.

In this paper we give the Lie point symmetry groups of a class of nonlinear diffusion equations.

First of all we consider the equation obtained by adding a nonlinear term to equation (1):

$$\partial u/\partial t = \Delta u + ku^q \tag{2}$$

where $q \in \mathbb{R}$. This equation appears when we include chemical reactions. It has been investigated in one space dimension by Steeb (1978).

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Theorem 2. The nonlinear diffusion equation (2) is invariant under the Lie point symmetry groups which are generated by the infinitesimal generators T, X_i, R_{ij} and

$$K = S - \frac{2}{(q-1)} u\,\partial/\partial u$$

where $i, j = 1, \ldots, n$.

Invariance of the nonlinear diffusion equation (2) under, say, K is equivalent to the statement that if u solves equation (2) then

$$V = [S + 2/(q-1)]u$$

solves the linearised equation

$$\partial V/\partial t = \Delta V + kqu^{q-1} V.$$

We mention that G_i ($i = 1, \ldots, n$) is a symmetry of equation (2) only if $q = 1$, in which case we have a linear equation.

Consider now the nonlinear diffusion equation

$$\partial u/\partial t = \operatorname{div}(u^p \operatorname{grad} u) \tag{3}$$

where $p \in \mathbb{R}$ ($p \neq 0$). Such nonlinear diffusion equations arise when we study concentration-dependent diffusion. We notice that equation (3) can be considered as a conservation law. For $n = 1$ the Lie point symmetry groups are given by Bluman and Kumei (1980). As far as we know only this case has been investigated so far. After rescaling the space coordinates we can write equation (3) as ($p \neq -1$)

$$\partial u/\partial t = \Delta(u^{p+1}). \tag{4}$$

Theorem 3. The nonlinear diffusion equation (4) is invariant under the Lie point symmetry groups which are generated by the infinitesimal generators T, X_i, R_{ij}, S and

$$Y = t\,\partial/\partial t - p^{-1} u\,\partial/\partial u.$$

If $p = -4/(n+2)$, then equation (4) also admits the symmetry generators

$$W_i = -2x_i \sum_{j=1}^{n} x_j\,\partial/\partial x_j + r^2\,\partial/\partial x_i + (n+2)x_i u\,\partial/\partial u.$$

It is worth mentioning that if $p = -2$ and $n = 1$, then equation (2) admits Lie Bäcklund transformation groups (Bluman and Kumei 1980).

Consider now the equation

$$\partial u/\partial t = \Delta(u^{p+1}) + ku^q \qquad (p \neq -1). \tag{5}$$

Theorem 4. The nonlinear diffusion equation (5) is invariant under the one-parameter Lie point symmetry groups generated by T, X_i, R_{ij} and

$$Z = S + \frac{2p}{(q-p-1)} Y.$$

If $q = p + 1$, then equation (4) is also invariant under Y.

We note that the infinitesimal generators given in theorem 1 form a basis of a non-Abelian Lie algebra and the same holds for the lists in theorems 2–4.

We now apply the above symmetries for finding similarity solutions to equations (4) and (5). Similarity solutions are available in abundance (see, for example, Bluman and Cole 1974, Boyer 1961, Grundy 1979, Crank 1975). However, the authors studied only the case where $n = 1$. We mention that Lie Bäcklund transformation groups (if any exist) can also be used for finding similarity solutions (Steeb and Oevel 1983).

Example 1. Let us consider equation (4). Taking a linear combination of the symmetries T, X_i $(i = 1, \ldots, n)$ we obtain the similarity *ansatz*

$$u(x_1, \ldots, x_n, t) = f(\zeta) \tag{6}$$

where the similarity variable is given by

$$\zeta = a_0 t + a_1 x_1 + \ldots + a_n x_n. \tag{7}$$

Inserting this *ansatz* into equation (4), we get the second-order nonlinear ordinary differential equation

$$f'' + pf^{-1}(f')^2 - \frac{a_0}{(p+1)|\boldsymbol{a}|^2} f^{-p} f' = 0 \tag{8}$$

where $f' \equiv \mathrm{d}f/\mathrm{d}\zeta$ and $|\boldsymbol{a}|^2 = a_1^2 + \ldots + a_n^2$.

To find a similarity solution to equation (4), we take a solution to equation (8), insert it into equation (6) and substitute the similarity variable given by equation (7).

Example 2. Again let us consider equation (4). The symmetry generator $\Sigma_i a_i X_i + pY$ is associated with the transformation group

$$x_1 = (x_1)_0 + a_1\varepsilon, \qquad \ldots, \qquad x_n = (x_n)_0 + a_n\varepsilon$$

$$t = t_0 \exp(p\varepsilon) \qquad u = u_0 \exp(-\varepsilon)$$

where ε is the group parameter. To find a similarity *ansatz* we set $x_0 = 0$, $t_0 = \zeta$. As a similarity variable ζ we find

$$\zeta = t \exp\left(-c \sum_{i=1}^{n} x_i\right).$$

Since $u = u_0 \exp(-\varepsilon)$ we get the similarity *ansatz*

$$u(x_1, \ldots, x_n, t) = f(\zeta) \exp\left(-c \sum_{i=1}^{n} x_i\right)$$

where $c \in \mathbb{R}$. Inserting this *ansatz* into equation (4) we get the nonlinear ordinary differential equation

$$np^2(p+1)c^2\zeta^2 f^p f'' + np^3(p+1)c^2\zeta^2 f^{p-1} f'^2$$
$$+ np(p+1)(3p+2)c^2\zeta f^p f' - f' + n(p+1)^2 c^2 f^{p+1} = 0.$$

Example 3. Consider the symmetry generator S of equation (4), which generates the transformation group

$$\boldsymbol{x} = \boldsymbol{x}_0 \exp(\varepsilon) \qquad t = t_0 \exp(2\varepsilon) \qquad u = u_0.$$

Choose $t_0 = \zeta$, and choose $\boldsymbol{x}_0$ so that $|\boldsymbol{x}_0|^2 = \zeta^2$. Then $\zeta = r^2/t$ and

$$u(x_1, \ldots, x_n, t) = f(\zeta).$$

The function f satisfies the nonlinear ordinary differential equation

$$f'' + pf^{-1}(f')^2 + \frac{n}{2\zeta}f' + \frac{1}{4(p+1)}f^{-p}f' = 0.$$

Let us now briefly describe how we can find conservation laws with the help of the symmetry generators. We study equation (3). Since equation (3) is itself a conservation law, we can apply an approach described recently (Steeb *et al* 1982). Within this approach we express the conservation law as a differential form. Then we calculate the Lie derivatives of this differential form with respect to the symmetry generators. The resulting differential forms are also conservation laws. Let us give an example. Consider the differential form

$$\omega = u\, dx_1 \wedge dx_2 \ldots \wedge dx_n + \sum_{k=1}^{n} (-1)^{k+1} u^p u_{x_k}\, dt \wedge dx_1 \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_n$$

where the hat denotes that this term is omitted. Let s be the mapping

$$s(\boldsymbol{x}, t) = (\boldsymbol{x}, t, u(\boldsymbol{x}, t));$$

then equation (3) can be written as

$$js^*(d\omega) = 0$$

where d denotes the exterior derivative and $\boldsymbol{x} = (x_1, \ldots, x_n)$. js is the jet extension of s. We note that $js^*(d\omega) = d(js^*\omega)$.

Let Q be a symmetry generator of equation (3) and $\bar{Q}$ the extended vector field of Q. Then according to the approach cited above we find that

$$js^*(L_{\bar{Q}}\omega) = 0$$

is a conservation law. When we use the symmetry generators given by theorem 2 we find that the vector fields T, X_i, Y, S and R_{ij} do not lead to new types of conservation laws. However, with the help of the symmetry generators W_i we find new conservation laws. Repeated application of W_i generates a hierarchy of conservation laws. Finally we mention that when we consider the linear diffusion equation (1) the infinitesimal generators G_i (Galilean transformation) and the infinitesimal generator C (conformal transformation) generate a hierarchy of conservation laws within the approach described above.

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